USE OF THE WKB METHOD TO INVESTIGATE CONNECTIVE HEAT TRANSFER IN A MICROPOLAR FLUID

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A general approximate solution is obtained for problems of heat transfer associated with a flow of micropolar fluid in a plane channel with boundary conditions of the first and second kind and its accuracy is determined.

There have been a large number of papers devoted to the solution of the problem of steady-state forced-convection heat transfer in channels. However, even in the case where several simplifying assumptions are made (the flow is assumed to be steady and stabilized, and the physical properties of the fluid constant; compressibility, energy dissipation, axial heat conduction, and also mass forces and moments, are neglected) it is difficult to obtain an analytical solution suitable for practical application. With the above assumptions the problem for cylindrical and plane channels reduces to the solution, with appropriate thermal boundary conditions, of the energy equation in the following form:

$$v(x_2)\frac{\partial T}{\partial x_1}=\frac{a}{x_2^{m-1}}\frac{\partial}{\partial x_2}\left(x_2^{m-1}\frac{\partial T}{\partial x_2}\right), \tag{1}$$

where

$$m=\begin{cases}1 & \text{for a plane channel,}\\ 2 & \text{for a cylindrical channel.}\end{cases}$$

Equation (1), satisfying prescribed boundary conditions, is usually solved by the method of separation of the variables, where

$$\Theta=Ce^{-\varepsilon^2\tilde{x}_1}\Psi(\tilde{x}_2) \tag{2}$$

and the problem reduces to finding the eigenvalues and eigenfunctions from the equation and boundary conditions for $\Psi(\tilde{x}_2)$, and also the constant coefficients C corresponding to them. In expression (2), Θ, $\tilde{x}_1$, and $\tilde{x}_2$ are the variables T, x_1, and x_2 in dimensionless form, and the form of which depends on the channel geometry and boundary conditions.

The solution of the equation for function $\Psi(\tilde{x}_2)$ is extended by representing it as a power series [1]. In this case, however, we do not obtain a solution in explicit form suitable for direct practical application. Moreover, in the investigation of heat transfer in a micropolar fluid, the hydrodynamics of which is largely determined by the microrotation of its particles [2], the indicated method leads to very laborious calculations. In [3] the asymptotic WKB method was used to construct the solution of Eq. (1) with thermal boundary conditions of the first kind for cases of flow of a Newtonian fluid in cylindrical and plane channels. In [4, 5] solutions were obtained for similar problems with thermal boundary conditions of the first and second kinds for a flow of micropolar fluid in channels. Smol'skii et al. [6] effectively used the WKB approximation to investigate heat transfer in viscoplastic media.

In papers devoted to the use of the WKB method for the solution of problems of type (1), however, the error associated with calculation of the main heat-transfer characteristics (the temperature T and heat flux on the wall q_w) was not estimated. Hence, the specific regions of the coordinate x_2 in which the use of the obtained expressions ensured minimum error were not indicated. We will consider these questions and will also construct a general solution of problems with thermal boundary conditions of the first and second kinds for the case of flow of a micropolar fluid in a plane channel.

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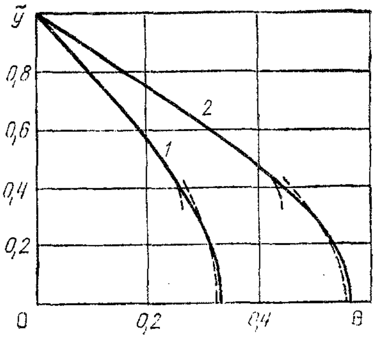

Fig. 1

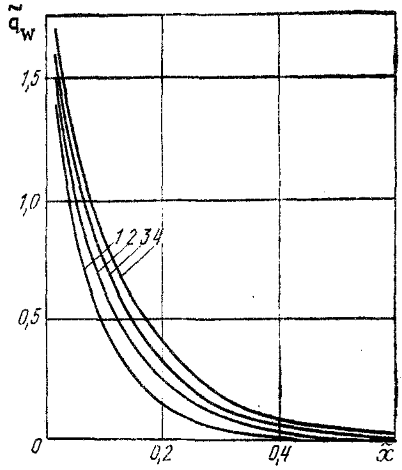

Fig. 2

Fig. 1. Cross-sectional temperature distribution for $\tilde{x}$ = 0.1: 1) δ = 0.83, k = 0.1; 2) δ = 0 (the continuous curves are the solution in the form of a power series, the dashed curves are the solution by the WKB method).

Fig. 2. Heat flux on wall along channel: 1) δ = 0.83, k = 0.1; 2) 0.83, 3; 3) 0.83, 8; 4) δ = 0.

Let an incompressible micropolar fluid flow under the action of a constant pressure gradient dp/dx between two stationary parallel plates, the distance between which is 2h. If the y axis of the Cartesian coordinate system is perpendicular to the channel planes, the nonzero components of the velocity vector $\vec{v}$ and microrotation vector $\vec{\nu}$ will be $v_x = v_x(y)$, $\nu_z = \nu_z(y)$.

Convective heat transfer in a micropolar fluid for particular cases of hydrodynamic boundary conditions for the vector $\vec{\nu}$ has been investigated. This involved the use of total no-slip conditions, when $\nu_z(\pm h) = 0$ on the stationary walls [4], also the conditions for absence of instantaneous stresses on the surfaces, when $(d\nu_z/dy)_{y=\pm h} = 0$ [5]. We consider more general hydrodynamic boundary conditions

$$v_x(\pm h) = 0,\ \nu_z(\pm h) = -\frac{\alpha}{2}\left(\frac{dv_x}{dy}\right)_{y=\pm h} \quad (0 \leqslant \alpha \leqslant 1), \tag{3}$$

for which the velocity profile is written as [7]

$$v_x = \frac{3}{2} v_m^N \left[1 - \frac{y^2}{h^2} + \delta \frac{\operatorname{cth} k}{k}\left(\frac{\operatorname{ch} ky/h}{\operatorname{ch} k} - 1\right)\right] = \frac{3}{2} v_m^N f(y/h). \tag{4}$$

Here

$$v_m^N = \frac{2h^2(-dp/dx)}{3(2\mu + \varkappa)};\quad \delta = \frac{2\varkappa(1-\alpha)}{2(\mu+\varkappa) - \varkappa\alpha};\quad k^2 = \frac{2\mu + \varkappa}{\mu + \varkappa}\ \frac{\varkappa}{\gamma}\ h^2.$$

We introduce the following dimensionless variables:

$$\tilde{x} = \frac{1}{\mathrm{Pe}}\ \frac{x}{2h},\quad \tilde{y} = \frac{y}{h}, \text{where } \mathrm{Pe} = \frac{2v_m^N h}{a},$$

$$\Theta = \begin{cases} \dfrac{T - T_w}{T_0 - T_w} & \text{for boundary condition} \quad T_w = \text{const}, \\[2ex] \dfrac{T - T_0}{2q_w h}\lambda & \text{for boundary condition} \quad q_w = \text{const}. \end{cases} \tag{5}$$

Substituting (4) in (1) with m = 1 and using (5), we obtain

$$\frac{3}{8} f(\tilde{y}) \frac{\partial\Theta}{\partial\tilde{x}} = \frac{\partial^2\Theta}{\partial\tilde{y}^2}. \tag{6}$$

We consider particular cases of thermal boundary conditions of the first kind when T_w = const

$$\Theta = 0,\ \tilde{x} > 0,\ \tilde{y} = \pm 1;\ \frac{\partial \Theta}{\partial \tilde{y}} = 0, \tag{7}$$
$$\tilde{x} > 0,\ \tilde{y} = 0;\ \Theta = 1,\ \tilde{x} = 0,\ -1 \leqslant \tilde{y} \leqslant 1$$

and of the second kind when q_w = const

$$\Theta(0, \tilde{y}) = 0,\ \left(\frac{\partial \Theta}{\partial \tilde{y}}\right)_{\tilde{y}=0} = 0,\quad \left(\frac{\partial \Theta}{\partial \tilde{y}}\right)_{\tilde{y}=1} = \frac{1}{2}. \tag{8}$$

The solution of problems of convective heat transfer in a micropolar fluid with the hydrodynamic boundary conditions (3) instead of their particular cases leads to final expressions similar to those found in [4] and [5], but with redetermined constants. Hence, a comparison of the solutions of Eq. (6) with thermal boundary conditions of the first [4] and second [5] kinds enables us to construct their general form:

$$\Theta(\tilde{x}, \tilde{y}) = g\Theta_{st}(\tilde{x}, \tilde{y}) + \sum_{n=g}^{\infty} C_n Y_n(\tilde{y}) \exp\left(-\frac{8}{3}\varepsilon_n^2 \tilde{x}\right), \tag{9}$$

$$Y_n(\tilde{y}) = M[f(\tilde{y})]^{-1/4} \cos\left(\varepsilon_n \int_0^{\tilde{y}} \sqrt{f(\xi)}\, d\xi\right) \quad \text{for} \quad 0 \leqslant \tilde{y} < 1, \tag{10}$$

$$Y_n(z) = \left(\frac{\pi \varepsilon_n z}{3}\right)^{1/2} M(-1)^n J_m\left(\frac{\varepsilon_n \sqrt{8\beta}}{3} z^{3/2}\right) \quad (z = 1 - \tilde{y}) \quad \text{for} \quad \tilde{y} \to 1, \tag{11}$$

$$C_n = (-1)^{n+g} b M^{-1} \varphi^{-1} \beta^{\frac{m}{2}} \varepsilon_n^{m\left(9g - \frac{7}{2}\right)},$$

$$\Theta_{st}(\tilde{x}, \tilde{y}) = 2\tilde{x}F^{-1} + \frac{3}{8}F^{-1}\left[(1-\tilde{\delta})\tilde{y}^2 - \frac{\tilde{y}^4}{6} + \delta\frac{2\,\mathrm{ch}\,k\tilde{y}}{k^2\,\mathrm{ch}\,k}\right] + c,$$

$$c = -\frac{9\tilde{\delta}}{8F^2}\left[\frac{13}{210\tilde{\delta}} + \frac{1}{k}\left(\frac{5}{12} - \frac{3\tilde{\delta}}{2k^2} - \frac{\tilde{\delta}}{2} - \frac{4}{k^4}\right)\mathrm{th}\,k\right.$$
$$\left. + \tilde{\delta}\left(\frac{1}{6} + \frac{1}{k^2}\right) + \frac{4}{k^4} - \frac{2}{3k^2} + \frac{\tilde{\delta}}{2k^2\,\mathrm{ch}^2\,k} - \frac{13}{60}\right],$$

$$\beta = 1 - \frac{\delta}{2},\quad \tilde{\delta} = \delta\frac{\mathrm{cth}\,k}{k},\quad F = 1 + \frac{3}{2}\delta\frac{1 - k\,\mathrm{cth}\,k}{k^2},$$

$$\varphi = \int_0^1 \sqrt{f(\tilde{y})}\, d\tilde{y},\quad M = \left(1 + \delta\frac{1 - \mathrm{ch}\,k}{k\,\mathrm{sh}\,k}\right)^{1/4},$$

$$g = 0,\ m = \frac{1}{3},\ b = 1.784,\ \varepsilon_n = \frac{\pi}{\varphi}\left(n + \frac{5}{12}\right) - T_w = \text{const},$$

$$g = 1,\ m = -\frac{1}{3},\ b = 0.971,\ \varepsilon_n = \frac{\pi}{\varphi}\left(n + \frac{1}{12}\right) - q_w = \text{const}.$$

From the obtained analytical solution it is relatively easy to calculate the temperature field in a micropolar fluid flowing in a plane channel. Expressions obtained with T_w = const are particularly suitable for calculation. Since in the given solution the expressions for $Y_n(\tilde{y})$ are asymptotic (the assumption $\varepsilon_n \to \infty$ was used), for practical application its accuracy has to be estimated. In addition, the regions of the cross section in which expressions (10) and (11) are most accurate must be determined. For this purpose problems (6), (7) and (6), (8) were solved by representation of function $Y(\tilde{y})$ in the form of a power series [1]

$$Y(\tilde{y}) = \sum_{m=0}^{\infty} b_{2m}(\varepsilon)\, \tilde{y}^{2m}. \tag{12}$$

The eigenvalues of the Sturm–Liouville problem for this function were found from the solution of the equations

$$\begin{aligned}&\sum_{m=0}^{\infty} b_{2m}(\varepsilon)=0 \quad \text{for boundary conditions (7),}\\ &\sum_{m=0}^{\infty} 2mb_{2m}(\varepsilon)=0 \quad \text{for boundary conditions (8),}\end{aligned} \tag{13}$$

in which the coefficients b_{2m} were determined from the following recurrence relations;

$$b_0=1,$$

$$b_2=-\frac{\varepsilon^2}{2}\left(1-\tilde{\delta}+\frac{\tilde{\delta}}{\operatorname{ch}k}\right),$$

$$b_4=-\frac{\varepsilon^2}{12}\left[b_2(1-\tilde{\delta})-b_0+\frac{\tilde{\delta}}{\operatorname{ch}k}\left(b_2+\frac{k^2}{2}b_0\right)\right],$$

$$\vdots$$

$$b_{2m+2}=-\frac{\varepsilon^2}{(2m+2)(2m+1)}\left[b_{2m}(1-\tilde{\delta})-b_{2m-2}+\frac{\tilde{\delta}}{\operatorname{ch}k}\sum_{p=0}^{m}\frac{k^{2p}}{(2p)!}b_{2m-2p}\right] \quad (m\geqslant 1).$$

The solution of Eqs. (13) gives us an infinite set of values of ε_n. Using the recurrence relations to substitute any value of ε_i in (12), we obtain the eigenfunction $Y_i(\tilde{y})$. It can be shown that, as for a Newtonian fluid, in this problem the coefficients C_n in (9) are determined from the expressions

$$C_n=-\frac{2}{\varepsilon_n\left(\dfrac{\partial Y}{\partial\varepsilon}\right)_{\varepsilon=\varepsilon_n,\ \tilde{y}=1}} \quad \text{for boundary conditions (7),}$$

$$C_n=\frac{1}{\varepsilon_n\left(\dfrac{\partial^2 Y}{\partial\varepsilon\partial\tilde{y}}\right)_{\varepsilon=\varepsilon_n,\ \tilde{y}=1}} \quad \text{for boundary conditions (8).}$$

The general solution of (9) was calculated in the ranges $0.02\leq\tilde{x}\leq 0.6$ and $0\leq\tilde{y}\leq 1$ on a Minsk-32 computer, which gave values of Θ correct to the fourth decimal place. A numerical comparison of the obtained solution with the asymptotic solution enables us to estimate the relative error δ_α associated with the use of the approximate WKB method. The considered ranges of the parameters characterizing the non-Newtonian behavior of a micropolar fluid are $k=0.1$–5 and $\delta=0.83$.

A numerical analysis of the two solutions shows that in the case where Eq. (11) is used to calculate Θ in the region $0.4\leq\tilde{y}\leq 1$ for $\tilde{x}\geq 0.05$, $\delta_\alpha<3\%$. When (10) is used in the region $0\leq\tilde{y}<0.4$ the error reaches 11%, but the introduction of the constant factor $s=0.921$ into this expression also leads in the calculation of Θ to $\delta_\alpha<3\%$. Thus, the use of (11) with $0.4\leq\tilde{y}\leq 1$ and (10) in the form

$$Y_n(\tilde{y})=sM[f(\tilde{y})]^{-1/4}\cos\left(\varepsilon_n\int_0^{\tilde{y}}\sqrt{f(\xi)}\,d\xi\right) \quad \text{for} \quad 0\leqslant\tilde{y}<0.4$$

ensures a value of $\delta_\alpha<3\%$. In the region $0.5\leq\tilde{y}\leq 1$ for $\tilde{x}\geq 0.05$ in the calculation of Θ by the WKB method $\delta_\alpha<0.5\%$. Figure 1 shows the curves for Θ obtained by the two methods.

In the case of boundary conditions of the first kind it is important to know the heat flux on the channel wall. The high accuracy of the asymptotic solution when $\tilde{y}\geq 0.5$ should lead to a small error in calculation of the heat flux on the wall. The WKB method leads to the following expression for $\tilde{q}_w=q_w h/\lambda(T_w-T_0)$:

$$\tilde{q}_w(\tilde{x})=1.591\varphi^{-1}\beta^{1/3}\sum_{n=0}^{\infty}\varepsilon_n^{-1/3}\exp\left(-\frac{8}{3}\varepsilon_n^2\tilde{x}\right). \tag{14}$$

A comparison of (14) with the solution obtained by representation of $Y_n(\tilde{y})$ as a power series shows that when the WKB method is used to calculate $\tilde{q}_w$ in the region $\tilde{x}>0.02$, $\delta_\alpha<1\%$. The series contained in (14) rapidly converges when $\tilde{x}>0.05$. For instance, if only the first two terms are taken, the error is less than 1%. Since calculation by formula (14) gives slightly overestimated values within the limits of this error, the use of only the first term of the series when $\tilde{x} > 0.05$ ensures that $\delta_\alpha < 1\%$. Hence, for practical calculations we can use the very simple formula

$$\tilde{q}_w(\tilde{x}) = 1.591 \frac{\sqrt[3]{\beta/\varepsilon_0}}{\varphi} \exp\left(-\frac{8}{3}\varepsilon_0^2\tilde{x}\right), \quad \varepsilon_0 = \frac{5\pi}{12\varphi}.$$

Figure 2 shows plots of $\tilde{q}_w$ against channel length, calculated from formula (14).

The curves corresponding to $\delta = 0$ represent the relations $\Theta(0.1;\ \tilde{y})$ and $\tilde{q}_w(\tilde{x})$, calculated without regard to the microstructure of the fluid. As the figures show, in certain conditions allowance for the self-rotation of the particles of the microstructured fluid in the calculation of the temperature difference $\Delta T = |T - T_w|$ and heat flux on the wall can lead to their considerable reduction. Similar results were obtained when Eq. (11) was solved for a flow of micropolar fluid in a cylindrical channel.

We can conclude from the above that, despite the asymptotic nature of the solution obtained by the WKB method, the calculated expressions for Θ and $\tilde{q}_w$ in the considered range of $\tilde{x}$ are fairly accurate over the entire cross section of the channel. This indicates that the WKB method is of great promise for the investigation of steady convective heat transfer in a micropolar fluid.

NOTATION

T_0 and T_w, temperatures of entrance section and wall of channel, respectively; dp/dx, pressure gradient; x_1, x_2, longitudinal and transverse coordinates, respectively (or x and y); $Pe = 2v_m^N h/a$, Peclet number; v_m^N, mean velocity of Newtonian fluid with viscosity $\mu + \varkappa/2$ in channel of width 2h; α, boundary condition parameter; 2h, width of channel; v_x and v_z, nonzero components of velocity and microrotation of micropolar fluid; a and λ, thermal diffusivity and thermal conductivity of fluid; $\varkappa$, μ, and γ, viscosities of micropolar fluid; q_w, heat flux density on wall; ε_n and $Y_n(\tilde{y})$, eigenvalues and eigenfunctions of Sturm—Liouville problem; C_n, constants that can be determined by using orthogonality of eigenfunctions.

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